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TECHNICAL REPORT

Office of Naval Research Contract No. N00014-86-K0029

WAITING TIMES FOR M/G/1 QUEUES WITH SERVICE-TIME-DEPENDENT SERVER VACATIONS

by

Percy H. Brill
and
Carl M. Harris

Report No. GMU/22474/110
June 22, 1989

Department of Operations Research and Applied Statistics
School of Information Technology and Engineering
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Waiting Times for M/G/1 Queues with Service-Time-Dependent Server Vacations

P.H. Brill and C.M. Harris

June 22, 1989

ABSTRACT

This paper shows how to determine the stationary distribution of the virtual wait in M/G/1 queues with either one at-a-time or exhaustive server vacations. Each vacation time may depend on the immediately preceding service time or on whether the server finds the system empty after returning from vacation. In this way, it is possible to model situations such as long service times followed by short vacations, and vice versa. The method of analysis employs level-crossing theory. Detailed examples are given for various cases of service and vacation-time distributions.

1 INTRODUCTION

Recent interest in queues with server vacations has focused on stochastic decomposition properties for the number-in-the-system random variable. Researchers who have investigated and obtained a number of important results for this aspect of the model include Gaver (1962), Cooper (1970), Levy and Yechiali (1975), Scholl and Kleinrock (1983), Fuhrmann (1984), Fuhrmann and Cooper (1985 a,b), Shanthikumar (1988), and Harris and Marchal (1988). Models have also been developed for G/G/1 server vacation queues in Doshi (1985), Keilson and Servi (1986), and Servi (1986). But few results are available for the stationary distribution of the waiting-time random variable in such queues. The mean and variance of the waiting time in the specialized case of D/G/1 queues with server vacation were given in Servi (1986) using complex transform methods. Doshi (1986) showed an integral equation for the probability density function (pdf) of the stationary virtual wait for an M/G/1 server-vacation model, based on Brill and Posner (1977, 1981). But his equation did not involve the kind of dependency considered in this paper, and Doshi also did not provide any of the detailed complexities necessary for carrying out such analyses.

A model with state-dependent server vacations was investigated by Harris and Marchal (1988). Their paper offered a decomposition principle for M/G/1 queues in which the server-vacation times depend on the number of customers remaining in the system at service completion epochs.

The object of the present paper is to derive the stationary distribution of the virtual waiting time in M/G/1 queues in which each server vacation time may depend on the immediately preceding service time. Consideration of this dependency allows, for example, the possibility of rewarding short service times with long vacations, penalizing long service times with short vacations, or modeling situations where the server naturally requires a service-time-dependent recovery period after each service. The modeling technique and solution method used in this paper are based on level crossing theory for obtaining probability distributions directly, originated by Brill (1975), and further elucidated, for example, in Brill and Posner (1977, 1981).

The level-crossing methodology is extremely advantageous in the present context, and yields a general integral equation for the pdf of the virtual wait immediately (given in Section 2, Equations (1) - (4)). This paper emphasizes how the general integral equation can be specialized in various important

cases of service-time and server-vacation-time distributions. It will become clear (Section 3) that this procedure may require careful analysis, according to the type of dependency of server vacation on customer service time. Furthermore, we obtain explicitly, or indicate how to obtain, the pdf of the virtual wait in these cases. The examples presented point to a potentially large class of models that can be solved in a similar manner.

Section 2 of this paper derives a general model equation for the stationary distribution of the virtual wait in M/G/1 queues with one-at-a-time server vacations depending on the immediately preceding service times. Variations of this equation are given for the exhaustive server vacation model and other special cases. Section 3 utilizes these model equations to obtain the stationary distribution of the virtual wait in four examples. These illustrative examples demonstrate an approach for obtaining the distribution of the virtual wait in other special cases of interest.

2 Model Equations for the Stationary Distribution of Virtual Wait in M/G/1 Queues with One-at-a-Time Server Vacations Depending on Service Times

Customers arrive in a Poisson stream at rate λ and are served in order of arrival by a single server who may take intermittent vacations following service completions. The service times are denoted by S_n, $n \geq 1$, having common probability distribution function $B(\cdot)$. Immediately following the nth service completion, the server takes a vacation of duration $V_n(S_n) \equiv V_n$, which may depend on S_n. Denote an arbitrary steady-state service time by $S \overset{dist}{=} S_n$, and a corresponding vacation time by $V \overset{dist}{=} V_n, n \geq 1$. If the server returns from a vacation and finds the system empty (no waiting customers), it immediately takes another vacation whose duration is denoted by V_0.

We define the conditional distribution function of the random variable V given S by

$$H(x,y) = P(V \leq x \mid S = y),$$

and the distribution function of V_0 by

$$H_0(x) = P(V_0 \leq x).$$

Also define $\overline{B}(x) = 1 - B(x), \overline{H}(x) = 1 - H(x)$, and $\overline{H}_0(x) = 1 - H_0(x)$, for all x.

Let $W(t)$ denote the virtual wait at time t, defined as the delay of a potential arrival at time t to the system until he begins service (at instant $W(t)+t$). The stationary distribution function and density function of $W(t)$ are denoted by

$$F(x) = \lim_{t \to \infty} P(W(t) \leq x)$$

and

$$f(x) = \frac{dF(x)}{dx},$$

respectively, assuming they exist. In this system all customers must wait a positive time with probability one, and thus $P(W(t) = 0) = 0, t > 0$, implying that $lim_{t \to \infty} P(W(t) = 0) = 0$. Although general conditions for ergodicity are not examined in this paper, these conditions are given for the special cases illustrated below in Section 3. We now employ system point-level crossing theory to derive the model equations for the pdf $f(w), w \geq 0$.

Level-crossing theory uses sample-path properties, and a sample path of the $\{W(t)\}$ process is shown in Figure 1. Without loss of generality, assume the system starts empty and the server begins a vacation at time $t = 0$. The sample path's slope is -1 for all $t \geq 0$, except at arrival epochs and the starts of empty system server vacations, which are epochs of jump discontinuity. At each arrival epoch, a jump is generated equal to the sum of the arrival's service time and the immediately subsequent server-vacation time.

From system point-level crossing theory, the long-run expected down-crossing rate of level $w > 0$ in the state space is equal to $f(w)$ (Brill 1975, or Brill & Posner 1977, 1981). The long-run expected upcrossing rate of level w is equal to the expression (for $w > 0$)

$$\lambda \int_{z=0}^{w} P(S + V > w - z) f(z)\, dz \; + \; f(0^+) P(V_0 > w)$$

$$= \quad \lambda \textstyle\int_{z=0}^{w} \left[\int_{y=0}^{w-z} \overline{H}(w - z - y, y)\, dB(y)\right] f(z) dz$$

$$+ \quad \lambda \textstyle\int_{z=0}^{w} \overline{B}(w - z) f(z)\, dz + f(0^+) \overline{H}_0(w),$$

recalling that $f(0^+)$ equals the long-run expected rate of level zero hits (Brill, 1975).

Equating the long-run expected rates of sample-path down- and up-crossings of state-space level $w > 0$ yields the model equation for the stationary pdf $f(w)$, $w > 0$:

$$\begin{aligned} f(w) &= \lambda \int_{z=0}^{w} \int_{y=0}^{w-z} \overline{H}(w-z-y,y) f(z) dB(y) dz \\ &+ \lambda \int_{z=0}^{w} \overline{B}(w-z) f(z) dz + f(0^+) \overline{H}_0(w). \end{aligned} \tag{1}$$

The normalizing condition is given by

$$\int_0^\infty f(w)\, dw = 1 \tag{2}$$

since the virtual wait is concentrated on $(0, \infty)$ with probability one.

If the vacation times $\{V_n\}$ are independent of the service times $\{S_n\}$ for all $n \geq 1$, then Equation (1) becomes (for $w > 0$)

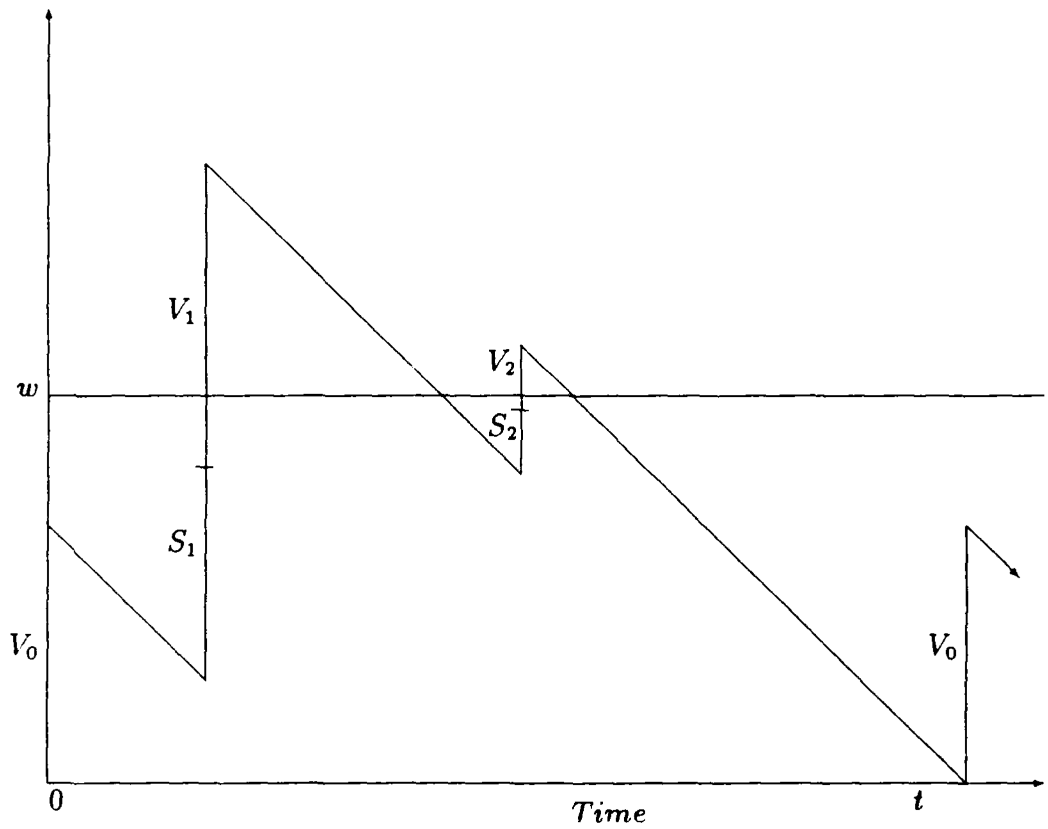
$$\begin{aligned} f(w) &= \lambda \int_{z=0}^{w} \int_{y=0}^{w-z} \overline{H}(w-z-y) f(z) dB(y) dz \\ &+ \lambda \int_{z=0}^{w} \overline{B}(w-z) f(z) dz + f(0^+) \overline{H}_0(w), \end{aligned} \tag{3}$$

where $\overline{H}(x) = P(V > x)$, $x \geq 0$.

In the exhaustive server-vacation problem, vacations occur only when the server finds no customers in the system, so that $V_n \equiv 0$, $n \geq 1$, and Equation (1) simplifies for $w > 0$ to

$$f(w) = \lambda \int_{z=0}^{w} \overline{B}(w-z) f(z) dz + f(0^+) \overline{H}_0(w). \tag{4}$$

Figure 1. A sample path of the process $\{W(t), t \geq 0\}$.



3 Examples

This section contains four examples which illustrate how to obtain the stationary distribution of the virtual wait. In the first example, service times have a generalized hyperexponential (GH) distribution (Botta, Harris & Marchal, 1987), and server-vacation times are exponentially distributed with means depending on the duration of the immediately preceding service times. In the second example, the service times are exponentially distributed and the vacation times are deterministic with value depending on the duration of the immediately preceding service times. Example 3 is a special case of Example 1 in which service times and server-vacation times are assumed independent. Finally, Example 4 illustrates an exhaustive server-vacation model which is a special case of Example 2.

Example 1

Assume that each service time is distributed as random variable S having the GH distribution with probability density

$$\frac{dP(S \leq x)}{dx} = \sum_{i=1}^{n} c_i\, e^{-\mu_i x} \qquad (x > 0),$$

where $\mu_i > 0$ and $-\infty < c_i < +\infty$ for $i = 1, \ldots, n$ ($n \geq 1$). Next assume that the server vacation times are exponential random variables with means $1/\gamma(S)$, where $\gamma(S)$ is a function of the immediately preceding service time S, and there exists some real number $A > 0$ such that

$$\gamma(S) = \begin{cases} \gamma_1 & \text{if } S \leq A \\ \gamma_2 & \text{if } S > A. \end{cases}$$

In this model, A may be considered to be a system control parameter.

Let V_0 be an exponential r.v. with mean $1/\gamma_0$. Then model Equation (1) for the stationary pdf of the virtual wait becomes

$$f(w) = \lambda \sum_{i=1}^{n} \int_{z=0}^{w} \int_{y=0}^{w-z} e^{-\gamma_1 (w-z-y)} c_i e^{-\mu_i y} f(z)\, dy\, dz$$

$$+ \lambda \sum_{i=1}^{n} (c_i/\mu_i) \int_{z=0}^{w} e^{-\mu_i (w-z)} f(z)\, dz + f(0^+) e^{-\gamma_0 w} \qquad (5)$$

$$(0 < w \leq A),$$

and

$$f(w) = \lambda \sum_{i=1}^{n} \int_{z=0}^{w-A} \left[\int_{y=0}^{A} e^{-\gamma_1 (w-z-y)} c_i e^{-\mu_i y} dy + \int_{y=A}^{w-z} e^{-\gamma_2 (w-z-y)} c_i e^{-\mu_i y} dy \right] f(z)\, dz$$

$$+ \lambda \sum_{i=1}^{n} \int_{z=w-A}^{w} \left[\int_{y=0}^{w-z} e^{-\gamma_1 (w-z-y)} c_i e^{-\mu_i y} dy \right] f(z)\, dz \qquad (6)$$

$$+ \lambda \sum_{i=1}^{n} \int_{z=0}^{w} (c_i/\mu_i) e^{-\mu_i (w-z)} f(z)\, dz + f(0^+) e^{-\gamma_0 w}$$

$$(A < w).$$

The pdf $f(w)$ is continuous for all $w > 0$ (Brill, 1987) and thus $f(A^-) = f(A^+)$.

Equations (5) and (6) can be rewritten respectively as

$$f(w) = \lambda \gamma_1 \sum_{i=1}^{n} \left[\frac{c_i}{\mu_i (\gamma_1 - \mu_i)} \right] \int_{z=0}^{w} e^{-\mu_i (w-z)} f(z)\, dz$$

$$- \lambda \sum_{i=1}^{n} \left[\frac{c_i}{\gamma_1 - \mu_i} \right] \int_{z=0}^{w} e^{-\gamma_1 (w-z)} f(z)\, dz + f(0^+) e^{-\gamma_0 w} \qquad (7)$$

$$(0 < w < A)$$

and

$$
\begin{aligned}
f(w) \;=\; & \lambda \sum_{i=1}^{n} \left[\frac{c_i e^{A(\gamma_1-\mu_i)}-1}{\gamma_1-\mu_i} \right] \int_{z=0}^{w-A} e^{-\gamma_1(w-z)} f(z)\,dz \\
& +\lambda \sum_{i=1}^{n} \left[\frac{c_i}{\gamma_2-\mu_i} \right] \int_{z=0}^{w-A} e^{-\mu_i(w-z)} f(z)\,dz \\
& -\lambda \sum_{i=1}^{n} \left[\frac{c_i e^{A(\gamma_2-\mu_i)}}{\gamma_2-\mu_i} \right] \int_{z=0}^{w-A} e^{-\gamma_2(w-z)} f(z)\,dz \\
& +\lambda \sum_{i=1}^{n} \left[\frac{c_i}{\gamma_1-\mu_i} \right] \int_{z=w-A}^{w} \left[e^{-\mu_i(w-z)} - e^{-\gamma_1(w-z)} \right] f(z)\,dz \\
& +\lambda \sum_{i=1}^{n} \frac{c_i}{\mu_i} \int_{z=0}^{w} e^{-\mu_i(w-z)} f(z)\,dz + f(0^+)\,e^{-\gamma_0 w} \\
& \qquad\qquad\qquad\qquad\qquad\qquad (A < w). \qquad (8)
\end{aligned}
$$

We can solve for $f(w)$ in (2), (7) and (8) by converting (7) to a differential equation, substituting its solution into (8), and applying (2). Alternatively, we may use numerical methods or level-crossing estimation (Brill, 1987).

Example 2

In this next example, there is a control parameter $A > 0$, and the vacation times are deterministic, depending on the preceding service times or whether the vacation originates when the system is empty. The service times, $\{S_n, n \geq 1\}$, are iid exponential random variables with mean $1/\mu$. Let S denote an arbitrary service time. If $S < A$, the server takes a vacation of duration $V = D_2$, and if $S > A$, the server gets a vacation of duration $D_1 < D_2$. Thus long service times ($S > A$) are followed by short vacations, and vice versa. Moreover, assume that vacations when the system is empty are $V_0 = D_0$, and that $0 < D_0 < D_1 < D_2 < A$ holds. Other cases for the value of A can be treated similarly. The model equations for $f(\cdot)$, the stationary pdf of the virtual wait, are as follows, noting that the function $f(w)$ has a discontinuity at $w = D_0$ (Brill, 1987):

$$f(w) = \lambda \int_{z=0}^{w} f(z)\,dz + f(0^+) \qquad (0 < w < D_0), \tag{9}$$

$$f(w) = \lambda \int_{z=0}^{w} f(z)\,dz \qquad (D_0 < w < D_2), \tag{10}$$

$$f(w) = \lambda \int_{z=w-D_2}^{w} f(z)\,dz + \lambda \int_{z=0}^{w-D_2} e^{-\mu(w-D_2-z)} f(z)\,dz \tag{11}$$

$$(D_2 < w < A + D_1),$$

$$f(w) = \lambda \int_{z=w-D_2}^{w} f(z)\,dz + \lambda \int_{z=w-A-D_1}^{w-D_2} e^{-\mu(w-D_2-z)} f(z)\,dz$$

$$+\,\lambda \int_{z=0}^{w-A-D_1} \left[e^{-\mu(w-D_1-z)} + e^{-\mu(w-D_2-z)} - e^{-\mu A}\right] f(z)\,dz \tag{12}$$

$$(A + D_1 < w < A + D_2),$$

$$f(w) = \lambda \int_{z=w-D_2}^{w} f(z)\,dz + \lambda \int_{z=w-A-D_1}^{w-D_2} e^{-\mu(w-D_2-z)} f(z)\,dz$$

$$+\,\lambda \int_{z=w-A-D_2}^{w-A-D_1} \left[e^{-\mu(w-D_1-z)} + e^{-\mu(w-D_2-z)} - e^{-\mu A}\right] f(z)\,dz \tag{13}$$

$$+\lambda \int_{z=0}^{w-A-D_2} e^{-\mu(w-D_1-z)} f(z)\,dz \qquad (A + D_2 < w).$$

We can solve for $f(w)$ in (2) and (9) - (13) by converting to differential equations, substituting the solutions for lower values of w into the equations for higher values, and finally applying (2). Thus, from (9) and (10), we get

$$f(w) = \begin{cases} a\,e^{\lambda w} & (0 < w < D_0) \\ b\,e^{\lambda w} & (D_0 \le D_2) \end{cases}$$

where a and b are constants to be determined. Balancing down- and up-crossing rates at state-space level D_0 yields $f(D_0^+) = \lambda \int_{z=0}^{D_0} f(z)\,dz$, resulting in $b = a(1 - e^{-\lambda D_0})$. This procedure can be continued in Equations (10) - (13), using the functional form of $f(w)$ for $0 < w < D_2$, etc. Alternatively, we may use numerical methods or level-crossing estimation to obtain $f(w), w > 0$.

Example 3

Consider a situation where the vacation times $\{V_n, n \geq 0\}$ are iid exponential random variables with mean $1/\eta$, and the service times are iid exponential random variables with mean $1/\mu$, $\mu \neq \eta$. The vacation and service times are independent, so that Equation (3) applies; thus the model equation for $f(\cdot)$ is

$$f(w) = \lambda \int_{z=0}^{w} \int_{y=0}^{w-z} e^{-\eta(w-z-y)} \mu e^{-\mu y} f(z)\,dy\,dz$$

$$+\lambda \int_{z=0}^{w} e^{-\mu(w-z)} f(z)\,dz + f(0^+)e^{-\eta w} \qquad (w > 0),$$

which simplifies to

$$f(w) = \frac{\lambda\eta}{\eta - \mu} \int_{z=0}^{w} e^{-\mu(w-z)} dz - \frac{\lambda\mu}{\eta - \mu} \int_{z=0}^{w} e^{-\eta(w-z)} f(z) dz$$
$$+ f(0^+)\,e^{-\eta w} \qquad (w > 0). \tag{14}$$

Equation (14) is a special case of (7) and (8) in Example 1, with $n = 1, \mu_1 \equiv \mu, \gamma_1 = \gamma_2 \equiv \eta$, letting $A = 0$ in (8) or letting $A \to \infty$ in (7).

Taking derivatives with respect to w in (14) and rearranging terms yields the linear, second-order differential equation

$$f''(w) + (\eta + \mu - \lambda)f'(w) + (\eta\mu - \lambda\eta - \lambda\mu)f(w) = 0. \tag{15}$$

We assume that the stationary distribution of the virtual wait exists, and hence the mean interarrival time exceeds the mean jump size for non-zero waits, as in the classical $M/G/1$ queue, that is, when

$$\frac{1}{\lambda} > \frac{1}{\mu} + \frac{1}{\eta}.$$

It follows that the characteristic equation of (15) has two real roots with positive product and negative sum, and hence both roots are negative. Denoting the roots by R_1 and R_2 yields the solution for the pdf of the virtual delay as

$$f(w) = a\, e^{R_1 w} + b\, e^{R_2 w} \qquad\qquad (w > 0),$$

where a and b are constants. Substituting this solution into (2) and equating the coefficient of $e^{-\mu w}$ on the right-hand side of identity (14) to zero yields

$$\begin{cases} \frac{a}{R_1} + \frac{b}{R_2} = -1 \\ \frac{a}{\mu+R_1} + \frac{b}{\mu+R_2} = 0 \end{cases}$$

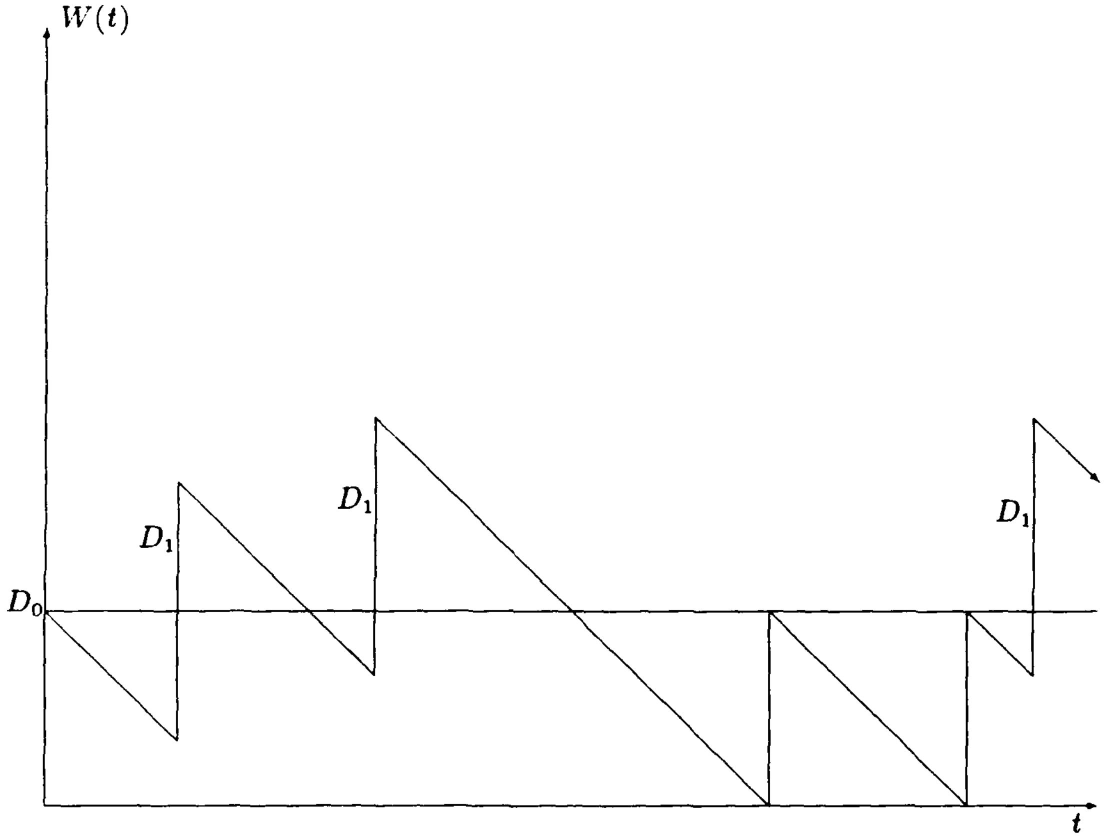
with solution

$$\begin{cases} a = \frac{(\eta\mu-\lambda\eta-\lambda\mu)(\mu+R_1)}{\mu(R_1-R_2)} \\ b = \frac{(\eta\mu-\lambda\eta-\lambda\mu)(\mu+R_2)}{\mu(R_2-R_1)}. \end{cases}$$

Example 4

This example treats a case of the exhaustive server-vacation M/G/1 model, so that $V_n \equiv 0$, $n \geq 1$. Assume that server-vacation times when the system is empty are deterministic, each equal to D_0, i.e., $V_0 = D_0$. Moreover, assume for the sake of exposition that service times are also deterministic, with each service $S = D_1$ and $D_1 > D_0$. (See Figure 2.)

Figure 2. Sample path in the exhaustive server-vacation model.



Equation (1) now becomes

$$\begin{aligned} f(w) &= \lambda \int_{z=w-D_1}^{w} f(z)\,dz \qquad (w > D_1) \\ f(w) &= \lambda \int_{z=0}^{w} f(z)\,dz \qquad (D_0 < w < D_1) \\ f(w) &= \lambda \int_{z=0}^{w} f(z)\,dz + f(0^+) \qquad (0 < w < D_0). \end{aligned}$$

For $w > D_1$, assume that $f(w) = ce^{\beta w}$. Then

$$ce^{\beta w} = \lambda \frac{c}{\beta} \left[e^{\beta w} - e^{\beta(w-D_1)} \right],$$

so that

$$\lambda e^{-\beta D_1} + \beta - \lambda = 0, \tag{16}$$

which yields the value of β, for $\beta < 0$, by the uniqueness of the solution of the integral equation.

For $w \in (D_0, D_1)$,

$$f'(w) = \lambda f(w),$$

so that

$$f(w) = d\,e^{\lambda w} \qquad (d = constant).$$

By continuity at D_1 (Brill, 1987), $f(D_1) = ce^{\beta D_1}$, and it follows that

$$d = e^{(\beta-\lambda)D_1} c. \tag{17}$$

For $w < D_0$,

$$f'(w) = \lambda f(w),$$

so that $f(w) = ae^{\lambda w}$ and $f(0^+) = a$ ($a = constant$). Note that there is a discontinuity at $w = D_0$ (Brill, 1987). However, $f(D_0^+)$ is the expected entrance rate to $[0, D_0)$; the expected exit rate from $[0, D_0)$ is

$$\lambda \int_{z=0}^{D_0} f(z)\, dz$$

and therefore

$$f(D_0^+) = \lambda \int_{z=0}^{D_0} f(z)\, dz.$$

Thus

$$c\left[e^{(\beta-\lambda)D_1}\right] e^{\lambda D_0} = \lambda \int_{z=0}^{D_0} a\, e^{\lambda z} dz,$$

and

$$a = c\left[\frac{exp[(\beta-\lambda)D_1 + \lambda D_0]}{exp(\lambda D_0) - 1}\right] \equiv cA. \tag{18}$$

Finally, from the normalizing condition (2), we get

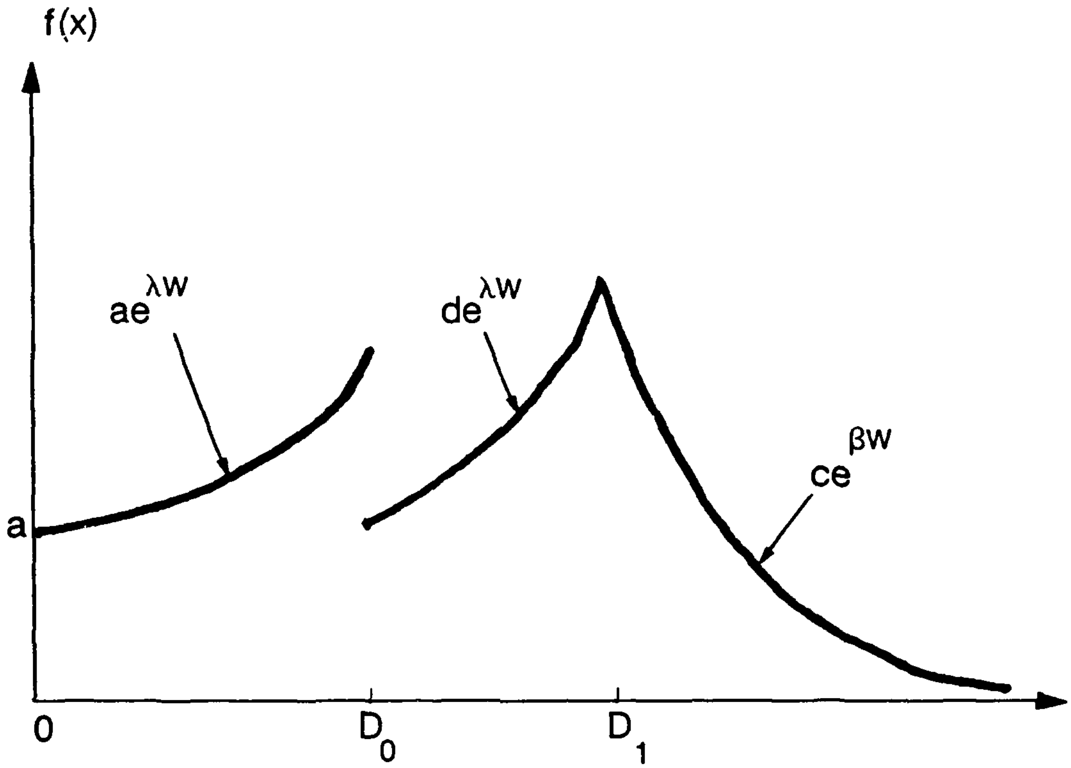
$$c\int_{w=D_1}^{\infty} e^{\beta w} dw + c\left[e^{(\beta-\lambda)D_1}\right]\int_{w=D_0}^{D_1} e^{\lambda w} dw + cA\int_{w=0}^{D_0} e^{\lambda w} dw = 1$$

or

$$c\left[-\frac{e^{\beta D_1}}{\beta} + \frac{e^{(\beta-\lambda)D_1}\left[e^{\lambda D_1} - e^{\lambda D_0}\right]}{\lambda} + A\frac{e^{\lambda D_0} - 1}{\lambda}\right] = 1. \tag{19}$$

The solution is obtained by substituting the values of c, a, d and β given in (19),(18), (17) and (16), respectively, into the formulas for the pdf $f(w)$ over the state space intervals $(0, D_0), (D_0, D_1)$ and (D_1, ∞).

Figure 3. PDF of virtual wait in the exhaustive server vacation model.



4 CONCLUDING REMARKS

Level-crossing theory is particularly useful for modeling and analyzing server-vacation queues, since service times and vacations are additive in the virtual waiting-time process. Moreover, the theory is concerned with balancing rates of down- and up-crossings, regardless of their qualitative origins.

Examples 1 and 2 (Section 3) indicate possible new ways to control such queues by treating the parameter A as a controllable. Control could be extended by using a collection of (positive integer) M such parameters $\{A_i\}$, where $0 < A_1 < A_2 < \ldots A_M$. Clearly, the examples of Section 3 demonstrate solution techniques that can be applied to a large class of $M/G/1$ server-vacation models, having various distributions of service and vacation times, with disciplines possibly other than first-come, first-served.

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